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An entanglement criterion for states in infinite-dimensional multipartite quantum systems

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In (Phys Lett A, 2002, 297: 4–8) an entanglement criterion for finite-dimensional bipartite systems is proposed: If ρ_{AB} is a separable state, then $\mathrm{Tr}(\rho_A^2) \geqslant \mathrm{Tr}(\rho^2)$ and $\mathrm{Tr}(\rho_B^2) \geqslant \mathrm{Tr}(\rho^2)$. In the present paper this criterion is extended to infinite-dimensional bipartite and multipartite systems. The reduction criterion presented in (Phys Rev A, 1999, 59: 4206–4216) is also generalized to infinite-dimensional case. Then it is shown that the former criterion is weaker than the later one.

quantum state, entanglement criterion, infinite-dimensional system

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1 Introduction

Quantum entanglement plays a crucial role in the rapidly developing theory of quantum information and quantum computation [1]. One of the most fundamental problem in entanglement theory is to determine whether a given quantum state is entangled or not. Nowadays, a number of different entanglement criteria have been found [2–18].

For the finite-dimensional bipartite quantum systems, Wu and Anandan [18] proposed a trace inequality criterion which reads as: If ρ is a separable state of a bipartite quantum system $A + B$, then $\mathrm{Tr}(\rho_A^2) \geqslant \mathrm{Tr}(\rho^2)$ and $\mathrm{Tr}(\rho_B^2) \geqslant \mathrm{Tr}(\rho^2)$, where ρ_A and ρ_B are the reduced density matrices of ρ, i.e. $\rho_A = \mathrm{Tr}_B(\rho)$ and $\rho_B = \mathrm{Tr}_A(\rho)$. Recall that, the reduction criterion says that if a state ρ is separable, then $\rho_A \otimes I_B - \rho \geqslant 0$ and $I_A \otimes \rho_B - \rho \geqslant 0$. It is claimed in [19] that the trace inequality criterion mentioned above is weaker than the reduction criterion. Though this claim is true, the proof of it is not right there.

The aim of this paper is to generalize the above two criteria to infinite-dimensional case and then show that the trace inequality criterion is weaker than the reduction criterion.

Recall that, a quantum state ρ (i.e. a positive trace-one operator) acting on a Hilbert space $H = H_A \otimes H_B$ with $\dim H_A \otimes H_B \leqslant +\infty$, is called *separable* if it can be written as a convex combination of some product states, i.e. ρ is of the form

$$\rho = \sum_i p_i \rho_i^A \otimes \rho_i^B, \quad \sum_i p_i = 1, \ p_i \geqslant 0, \tag{1}$$

or, it can be approximated in the trace norm by the states of the form in eq. (1) [20], where ρ_i^A and ρ_i^B are (pure) states in the subsystems H_A and H_B, respectively. Otherwise, ρ is called *entangled*. A quantum state ρ (i.e. a positive trace-one operator) acting on a Hilbert space $H = H_1 \otimes H_2 \otimes \ldots \otimes H_m$ with $\dim H_1 \otimes H_2 \otimes \ldots \otimes H_m \leqslant +\infty$ is defined to be *fully separable* if it has the form

$$\rho = \sum_i p_i \rho_i^{(1)} \otimes \rho_i^{(2)} \otimes \ldots \otimes \rho_i^{(m)}, \quad \sum_i p_i = 1, \ p_i \geqslant 0, \tag{2}$$

or it can be approximated in the trace norm by the states of the above form, where $\rho_i^{(k)}$s are (pure) states in the subsystems H_k, $k = 1, 2, \ldots, m$.

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We fix some notations used in this paper. In quantum mechanics, a quantum system is associated with a separable complex Hilbert space H, i.e. the state space. Let H_A, H_B, etc., be complex separable Hilbert spaces (associated with quantum systems), throughout the paper we use the Dirac's symbols. The brackets notation, $\langle\cdot|\cdot\rangle$ stand for the inner product in the given Hilbert spaces. The set of all bounded linear operators on some Hilbert space H is denoted by $B(H)$. Let $S(H_A \otimes H_B)$ be the set of all states in $H_A \otimes H_B$. It is obvious that $S(H_A \otimes H_B) \subseteq B(H_A \otimes H_B)$. By $\mathcal{T}(H)$ we denote the set of all trace class operators of Hilbert spaces H. If $T \in \mathcal{T}(H)$, we have $\|T\|_{Tr} = Tr((T^\dagger T)^{\frac{1}{2}}) < +\infty$, where $\|\cdot\|_{Tr}$ denote the trace norm. Recall that, if a quantum system is in one of a number of states $|\psi_i\rangle$, where i is an index, with respective probabilities p_i, then $\{p_i, |\psi_i\rangle\}$ is called an ensemble of pure states, and the associated density operator for the system is defined by $\rho = \sum_i p_i|\psi_i\rangle\langle\psi_i|$. Obviously, ρ is a pure state if and only if $Tr(\rho^2) = 1$.

2 The main results

In this section we present the main results and their proofs.

Our first result is an infinite-dimensional version of the trace inequality criterion proposed in [18].

Theorem 2.1 Let H_A, H_B be complex separable Hilbert spaces with $\dim(H_A \otimes H_B) = +\infty$. If $\rho \in S(H_A \otimes H_B)$ is separable, then $\mathrm{Tr}(\rho_A^2) \geqslant \mathrm{Tr}(\rho^2)$ and $\mathrm{Tr}(\rho_B^2) \geqslant \mathrm{Tr}(\rho^2)$, where $\rho_A = \mathrm{Tr}_B(\rho)$, $\rho_B = \mathrm{Tr}_A(\rho)$.

In order to prove this theorem, we need a lemma.

Lemma 2.2 Let H be a complex Hilbert space, $T_n, T \in \mathcal{T}(H)$. Then $\lim_{n\to+\infty} T_n = T$ under the trace norm implies that $\lim_{n\to+\infty} T_n^2 = T^2$ under the trace norm.

Proof. Note that, for $A \in \mathcal{T}(H)$, we have $\|A\| \leqslant \|A\|_{\mathrm{Tr}} < +\infty$. As $T_n, T \in \mathcal{T}(H) \subseteq B(H)$ and $\|T_n - T\|_{\mathrm{Tr}} \to 0$, there exists some positive number M such that $\sup_n\{\|T_n\|_{\mathrm{Tr}}\} = M < +\infty$. Then, it follows from

$$\begin{aligned}
&\|T_n^2 - T^2\|_{\mathrm{Tr}} \\
&= \|T_n^2 - T_nT + T_nT - T^2\|_{\mathrm{Tr}} \\
&\leqslant M \cdot \|T_n - T\|_{\mathrm{Tr}} + \|T\| \cdot \|T_n - T\|_{\mathrm{Tr}}
\end{aligned}$$

that

$$\|T_n^2 - T^2\|_{\mathrm{Tr}} \to 0 \text{ as } n \to +\infty.$$

Proof of Theorem 2.1. Let $S_{S-p}(H_A \otimes H_B)$ be the set of all separable pure states. If ρ is separable, then it admits a representation of Bochner integral [21]

$$\rho = \int_{S_{S-p}} \varphi(\rho^A \otimes \rho^B)\mathrm{d}\mu(\rho^A \otimes \rho^B), \tag{3}$$

where μ is a Borel probability measure on $S_{S-p}(H_A \otimes H_B)$, $\rho^A \otimes \rho^B \in S_{S-p}(H_A \otimes H_B)$, and $\varphi : S_{S-p} \to S_{S-p}$ is a measurable function. It follows that there exists a sequences of step function φ_n, such that

$$\varphi(\rho^A \otimes \rho^B) = \lim_{n\to+\infty} \varphi_n(\rho^A \otimes \rho^B)$$

with respect to the trace norm, where

$$\varphi_n(\rho^A \otimes \rho^B) = \sum_{i=1}^{k_n} \mathcal{X}_{E_i}(\rho^A \otimes \rho^B)\rho_i^A \otimes \rho_i^B,$$

and $\mathcal{X}_{E_i}(\cdot)$ is the characteristic function of E_i, $\{E_i\}_{i=1}^{k_n}$ is a partition of $S_{S-p}(H_A \otimes H_B)$. Denote by E the set of all possible partitions $\{E_i\}_{i=1}^{k_n}$ of $S_{S-p}(H_A \otimes H_B)$. Then E is a direct set and we have

$$\rho = \lim_{\{E_i\}\in E} \sum_i \mu(E_i)\rho_i^A \otimes \rho_i^B, \tag{4}$$

with respect to the trace norm, as well as with respect to the Hilbert Schmidt norm, where ρ_i^A and ρ_i^B are pure states respectively in H_A and H_B, i.e. there exist unit vectors $\{|\psi_i\rangle\}$ in H_A and $\{|\phi_i\rangle\}$ in H_B, respectively, such that $\rho_i^A = |\psi_i\rangle\langle\psi_i|$ and $\rho_i^B = |\phi_i\rangle\langle\phi_i|$. It is well known that Tr is a completely positive linear functional of $\mathcal{T}(H)$ with $\dim H \leqslant +\infty$. In fact, taking any orthonormal basis $\{|e_i\rangle\}_{i=1}^{\dim H}$ of H, the complete positivity of Tr comes from the fact $\mathrm{Tr}(T)(|e_1\rangle\langle e_1|) = \sum_{i=1}^{\infty} E_iTE_i^\dagger$, where $E_i = |e_1\rangle\langle e_i|$. Thus Tr is completely bounded and hence $I\otimes\mathrm{Tr}$ is continuous. So we have

$$\rho_A = \mathrm{Tr}_B(\rho) = \lim_{\{E_i\}\in E} \sum_i \mu(E_i)\rho_i^A, \tag{5}$$

with respect to the trace norm. By Lemma 2.2, one gets

$$\begin{aligned}
\mathrm{Tr}(\rho_A^2) &= \lim_{\{E_i\}\in E} \mathrm{Tr}([\sum_i \mu(E_i)\rho_i^A]^2) \\
&= \lim_{\{E_i\}\in E} \mathrm{Tr}(\sum_i \sum_j \mu(E_i)\mu(E_j)\rho_i^A\rho_j^A) \\
&= \lim_{\{E_i\}\in E} \mathrm{Tr}(\sum_i \sum_j \mu(E_i)\mu(E_j)|\psi_i^A\rangle\langle\psi_i^A| \cdot |\psi_j^A\rangle\langle\psi_j^A|) \\
&= \lim_{\{E_i\}\in E} [\sum_i \sum_j \mu(E_i)\mu(E_j)|\langle\psi_i^A|\psi_j^A\rangle|^2] \\
&\geqslant \lim_{\{E_i\}\in E} [\sum_i \sum_j \mu(E_i)\mu(E_j)|\langle\psi_i^A|\psi_j^A\rangle\langle\phi_i^B|\phi_j^B\rangle|^2].
\end{aligned}$$

On the other hand, by Lemma 2.2, we have

$$\begin{aligned}
\mathrm{Tr}(\rho^2) &= \lim_{\{E_i\}\in E} \mathrm{Tr}([\sum_i \mu(E_i)\rho_i^A \otimes \rho_i^B]^2) \\
&= \lim_{\{E_i\}\in E} \mathrm{Tr}(\sum_i \sum_j \mu(E_i)\mu(E_j)\rho_i^A \otimes \rho_i^B\rho_j^A \otimes \rho_j^B) \\
&= \lim_{\{E_i\}\in E} [\sum_i \sum_j \mu(E_i)\mu(E_j)|\langle\psi_i^A|\psi_j^A\rangle\langle\phi_i^B|\phi_j^B\rangle|^2].
\end{aligned}$$

So we obtain that $\mathrm{Tr}(\rho_A^2) \geqslant \mathrm{Tr}(\rho^2)$. The inequality $\mathrm{Tr}(\rho_B^2) \geqslant \mathrm{Tr}(\rho^2)$ can be checked similarly. This completes the proof.

We give an example to illustrate how to apply the trace inequality criterion established in Theorem 2.1.

Example Let $H = H_A \otimes H_B$ be a Hilbert space with $\dim H_A = 2$ and $\dim H_B = +\infty$. We consider a state $\rho \in S(H_A \otimes H_B)$ with the following form:

$$\rho = \frac{x}{2}|00'\rangle\langle 00'| + \frac{1-x}{2}(|01'\rangle + |10'\rangle)(\langle 01'| + \langle 10'|) + \frac{x}{2}(|02'\rangle + |11'\rangle)(\langle 02'| + \langle 11'|),$$

where $0 \leqslant x \leqslant 1$, $\{|0\rangle, |1\rangle\}$ is an orthonormal basis of H_A and $\{|0'\rangle, |1'\rangle, \cdots\}$ is an orthonormal basis of H_B. Then

$$\rho = \left(\begin{array}{cccccc|cccccc}
\frac{x}{2} & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots \\
0 & \frac{1-x}{2} & 0 & \cdots & 0 & \cdots & \frac{1-x}{2} & 0 & 0 & \cdots & 0 & \cdots \\
0 & 0 & \frac{x}{2} & \cdots & 0 & \cdots & 0 & \frac{x}{2} & 0 & \cdots & 0 & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots \\
0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \\
\hline
0 & \frac{1-x}{2} & 0 & \cdots & 0 & \cdots & \frac{1-x}{2} & 0 & 0 & \cdots & 0 & \cdots \\
0 & 0 & \frac{x}{2} & \cdots & 0 & \cdots & 0 & \frac{x}{2} & 0 & \cdots & 0 & \cdots \\
0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots \\
0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots
\end{array}\right)$$

and $\rho_A = \begin{pmatrix} \frac{1+x}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$. It is obvious that

$$\mathrm{Tr}(\rho^2) = \frac{9}{4}x^2 - 2x + 1 \quad \text{and} \quad \mathrm{Tr}(\rho_A^2) = \frac{1}{4}(x^2 + 2x + 2).$$

Thus $\mathrm{Tr}(\rho_A^2) < \mathrm{Tr}(\rho^2)$ whenever $0 \leqslant x < \frac{1}{4}$. So ρ is entangled whenever $0 \leqslant x < \frac{1}{4}$.

For finite-dimensional case, the trace inequality criterion in [18] is valid for multipartite systems. Namely, if ρ is a fully separable state acting on $H_1 \otimes H_2 \otimes \cdots \otimes H_m$, then

$$\mathrm{Tr}(\rho_{\alpha_1}^2) \geqslant \mathrm{Tr}(\rho_{\alpha_1\alpha_2}^2) \geqslant \cdots \geqslant \mathrm{Tr}(\rho_{\alpha_1\alpha_2\cdots\alpha_r}^2) \geqslant \cdots \geqslant \mathrm{Tr}(\rho^2), \quad (6)$$

where $\alpha_1\alpha_2\cdots\alpha_r$ represent r distinct elements from the set $\{1, 2, \cdots, m\}$, and ρ_{α_1} is the reduced density matrix of the subsystem α_1, $\rho_{\alpha_1\alpha_2}$ is the reduced density operator of the system $\alpha_1 + \alpha_2$, etc., that is, ρ_{α_1} is obtained by tracing over all subsystems except α_1, $\rho_{\alpha_1\alpha_2}$ is obtained by tracing over all subsystems except α_1 and α_2, and so on. In what follows, we will generalize Theorem 2.1 to multipartite cases.

Let $H_1, H_2, \ldots, H_m$ be separable complex Hilbert spaces with $\dim(H_1 \otimes H_2 \otimes \cdots \otimes H_m) = +\infty$. By $S_{S-p}(H_1 \otimes H_2 \otimes \cdots \otimes H_m)$ we denote the set of all full separable pure states acting on $H_1 \otimes H_2 \otimes \cdots \otimes H_m$. According to [21], we have that if $\rho \in S(H_1 \otimes H_2 \otimes \cdots \otimes H_m)$ is fully separable, then ρ admits a representation of Bochner integral

$$\rho = \int_{S_{S-p}} \varphi(\rho^{(1)} \otimes \rho^{(2)} \otimes \cdots \otimes \rho^{(m)}) d\mu(\rho^{(1)} \otimes \rho^{(2)} \otimes \cdots \otimes \rho^{(m)}), \quad (7)$$

where μ is a Borel probability measure on $S_{S-p}(H_1 \otimes H_2 \otimes \cdots \otimes H_m)$, $\rho^{(1)} \otimes \rho^{(2)} \otimes \cdots \otimes \rho^{(m)} \in S_{S-p}(H_1 \otimes H_2 \otimes \cdots \otimes H_m)$, and $\varphi : S_{S-p} \to S_{S-p}$ is a measurable function. It follows that there exists a sequences of step function φ_n such that

$$\varphi(\rho^{(1)} \otimes \rho^{(2)} \otimes \cdots \otimes \rho^{(m)}) = \lim_{n\to\infty} \varphi_n(\rho^{(1)} \otimes \rho^{(2)} \otimes \cdots \otimes \rho^{(m)})$$

with respect to the trace norm, where

$$\varphi_n(\rho^{(1)} \otimes \rho^{(2)} \otimes \cdots \otimes \rho^{(m)}) = \sum_{i=1}^{k_n} \mathcal{X}_{E_i}(\rho^{(1)} \otimes \rho^{(2)} \otimes \cdots \otimes \rho^{(m)}) \times \rho_i^{(1)} \otimes \rho_i^{(2)} \otimes \cdots \otimes \rho_i^{(m)},$$

$\mathcal{X}_{E_i}(\cdot)$ is the characteristic function of E_i, and $\{E_i\}_{i=1}^{k_n}$ is a partition of $S_{S-p}(H_1 \otimes H_2 \otimes \cdots \otimes H_m)$. Let us denote by E the set of all finite partitions $\{E_i\}_{i=1}^{k_n}$ of $S_{S-p}(H_1 \otimes H_2 \otimes \cdots \otimes H_m)$. Now we have

$$\rho = \lim_{\{E_i\}\in E} \sum_i \mu(E_i) \rho_i^{(1)} \otimes \rho_i^{(2)} \otimes \cdots \otimes \rho_i^{(m)} \quad (8)$$

with respect to the trace norm, as well as with respect to the Hilbert Schmidt norm, where $\rho_i^{(j)} = |\psi_i^j\rangle\langle\psi_i^j|$ are pure states in H_j, $j = 1, 2, \ldots, m$.

Theorem 2.3 Let $H_1, H_2, \cdots, H_m$ be separable complex Hilbert spaces with $\dim H_1 \otimes H_2 \otimes \cdots \otimes H_m = +\infty$. If $\rho \in S(H_1 \otimes H_2 \otimes \cdots \otimes H_m)$ is a fully separable state, then

$$\mathrm{Tr}(\rho_{\alpha_1}^2) \geqslant \mathrm{Tr}(\rho_{\alpha_1\alpha_2}^2) \geqslant \cdots \geqslant \mathrm{Tr}(\rho_{\alpha_1\alpha_2\cdots\alpha_r}^2) \geqslant \cdots \geqslant \mathrm{Tr}(\rho^2), \quad (9)$$

where $\alpha_1\alpha_2\cdots\alpha_r$ represent r distinct elements from the set $\{1, 2, \ldots, m\}$, and ρ_{α_1} is the reduced density operator of the system α_1, $\rho_{\alpha_1\alpha_2}$ is the reduced density operator of the system $\alpha_1 + \alpha_2$, etc., that is, ρ_{α_1} is obtained by tracing over all subsystems except α_1, $\rho_{\alpha_1\alpha_2}$ is obtained by tracing over all subsystems except α_1 and α_2, and so on.

Proof. Since Tr is a completely bounded functional, $\rho_{\alpha_1\alpha_2\cdots\alpha_r}$ is well-defined. From eq. (8), we can see that for every integer r satisfying $1 \leqslant r \leqslant m$, $\rho_{\alpha_1\alpha_2\cdots\alpha_r}$ has a representation as follows:

$$\rho_{\alpha_1\alpha_2\cdots\alpha_r} = \lim_{\{E_i\}\in E} \sum_i \mu(E_i) \rho_i^{(\alpha_1)} \otimes \rho_i^{(\alpha_2)} \otimes \cdots \otimes \rho_i^{(\alpha_r)}.$$

By Lemma 2.2, we get

$$\begin{aligned}
&\mathrm{Tr}(\rho_{\alpha_1\alpha_2\cdots\alpha_r}^2) \\
&= \lim_{\{E_i\}\in E} \mathrm{Tr}\Big(\sum_{i,j} \mu(E_i)\mu(E_j)(\rho_i^{(\alpha_1)} \otimes \rho_i^{(\alpha_2)} \otimes \cdots \otimes \rho_i^{(\alpha_r)}) \\
&\quad \times (\rho_j^{(\alpha_1)} \otimes \rho_j^{(\alpha_2)} \otimes \cdots \otimes \rho_j^{(\alpha_r)})\Big) \\
&= \lim_{\{E_i\}\in E} \Big[\sum_i \sum_j \mu(E_i)\mu(E_j) |\langle \psi_i^{\alpha_1}\psi_i^{\alpha_2}\cdots\psi_i^{\alpha_r} | \psi_j^{\alpha_1}\psi_j^{\alpha_2}\cdots\psi_j^{\alpha_r}\rangle|^2 \Big]
\end{aligned}$$

$$\geqslant \lim_{\{E_i\}\in E}\Big[\sum_i\sum_j \mu(E_i)\mu(E_j)|\langle\psi_i^{\alpha_1}\psi_i^{\alpha_2}\cdots\psi_i^{\alpha_r}\psi_i^{\alpha_{r+1}}|$$

$$\psi_j^{\alpha_1}\psi_j^{\alpha_2}\cdots\psi_j^{\alpha_r}\psi_j^{\alpha_{r+1}}\rangle|^2\Big]$$

$$= \mathrm{Tr}(\rho^2_{\alpha_1\alpha_2\cdots\alpha_{r+1}}).$$

So we have

$$\mathrm{Tr}(\rho^2_{\alpha_1}) \geqslant \mathrm{Tr}(\rho^2_{\alpha_1\alpha_2}) \geqslant \cdots \geqslant \mathrm{Tr}(\rho^2_{\alpha_1\alpha_2\cdots\alpha_r}) \geqslant \cdots \geqslant \mathrm{Tr}(\rho^2). \quad \square$$

Next, we extend the reduction criterion to infinite-dimensional case.

Theorem 2.4 Let $\rho \in S(H_A \otimes H_B)$ with $\dim(H_A \otimes H_B) = +\infty$. If ρ is a separable state, then

$$\rho_A \otimes I_B - \rho \geqslant 0, \quad I_A \otimes \rho_B - \rho \geqslant 0. \qquad (10)$$

Proof. We only need to prove the inequality $\rho_A \otimes I_B - \rho \geqslant 0$. Another one can be checked similarly. Since ρ is separable, by eqs. (4) and (5), we have

$$\rho = \lim_{\{E_i\}\in E}\sum_i \mu(E_i)\rho_i^A \otimes \rho_i^B,$$

and

$$\rho_A = \mathrm{Tr}_B(\rho) = \lim_{\{E_i\}\in E}\sum_i \mu(E_i)\rho_i^A$$

with respect to the trace norm. Then

$$\begin{aligned}
& \rho_A \otimes I_B - \rho \\
= \;& \lim_{\{E_i\}\in E}\big(\textstyle\sum_i \mu(E_i)\rho_i^A \otimes I_B - \sum_i \mu(E_i)\rho_i^A \otimes \rho_i^B\big) \\
= \;& \lim_{\{E_i\}\in E}\big[\textstyle\sum_i \mu(E_i)(\rho_i^A \otimes I_B - \rho_i^A \otimes \rho_i^B)\big] \\
= \;& \lim_{\{E_i\}\in E}\big[\textstyle\sum_i \mu(E_i)(\rho_i^A \otimes (I_B - \rho_i^B))\big] \\
\geqslant \;& 0
\end{aligned}$$

as desired.

Now we discuss the relation between the trace inequality criterion and the reduction criterion.

Definition 2.5 Let (a) and (b) be two necessary separability criteria. By $\varepsilon(a)$ and $\varepsilon(b)$, we denote the set of entangled states detected by (a) and (b), respectively. We say that (a) is weaker than (b) if $\varepsilon(a) \subseteq \varepsilon(b)$. In this case we also say that (b) is stronger than (a). We say that (a) and (b) are equivalent if $\varepsilon(a) = \varepsilon(b)$. Finally we say that (a) and (b) are independent if (a) is neither weaker nor stronger than (b).

In [19], it is pointed out that the trace inequality criterion is weaker than the reduction criterion for the finite-dimensional systems. However the proof provided there is not correct since the argument is based on an incorrect equation (see eq. (3) in [19], it is stated there that "... $\sqrt{\rho} = (\sqrt{\rho_1} \otimes I)R^\dagger = R(\sqrt{\rho_1}) \otimes I$. Therefore, $\rho = R^\dagger(\sqrt{\rho_1} \otimes I)R$ and ...". This is not correct since $\rho = R(\sqrt{\rho_1} \otimes I)R^\dagger \neq R^\dagger(\sqrt{\rho_1} \otimes I)R$. The fact is true that the trace inequality criterion is weaker than the reduction criterion for systems of any dimension. We give a correct proof that is valid for both finite-dimensional and infinite-dimensional cases.

Theorem 2.6 The trace inequality criterion is weaker than the reduction criterion for both finite- and infinite-dimensional bipartite systems.

Proof. Assume that $\dim H_A \otimes H_B \leqslant +\infty$. Let $\rho \in S(H_A \otimes H_B)$ be any state satisfying $\rho_A \otimes I_B - \rho \geqslant 0$. Since the square root is an operator monotone function, we have $\sqrt{\rho_A} \otimes I_B - \sqrt{\rho} \geqslant 0$. By Douglas theorem [22], there exists a contractive operator R such that $\sqrt{\rho} = (\sqrt{\rho_A} \otimes I_B)R = R^\dagger(\sqrt{\rho_A} \otimes I_B)$. It follows that

$$\rho = (\sqrt{\rho_A} \otimes I_B)RR^\dagger(\sqrt{\rho_A} \otimes I_B).$$

Thus,

$$\rho_A = Tr_B(\rho) = \sqrt{\rho_A}\mathrm{Tr}_B(RR^\dagger)\sqrt{\rho_A},$$

which implies that

$$\mathrm{Tr}_B(RR^\dagger)\sqrt{\rho_A} = \sqrt{\rho_A} = \sqrt{\rho_A}\mathrm{Tr}_B(RR^\dagger).$$

Consequently,

$$\rho_A^2\mathrm{Tr}_B(RR^\dagger) = \rho_A^2.$$

Since

$$\rho^2 = \sqrt{\rho}\rho\sqrt{\rho} = R^\dagger(\rho_A \otimes I_B)RR^\dagger(\rho_A \otimes I_B)R \leqslant R^\dagger(\rho_A^2 \otimes I_B)R,$$

one has

$$\begin{aligned}
\mathrm{Tr}(\rho^2) &\leqslant \mathrm{Tr}(R^\dagger(\rho_A^2 \otimes I_B)R) \\
&= \mathrm{Tr}((\rho_A^2 \otimes I_B)RR^\dagger) \\
&= \mathrm{Tr}_A(\mathrm{Tr}_B((\rho_A^2 \otimes I_B)RR^\dagger)) \\
&= \mathrm{Tr}_A(\rho_A^2\mathrm{Tr}_B(RR^\dagger)) \\
&= \mathrm{Tr}_A(\rho_A^2) \\
&= \mathrm{Tr}(\rho_A^2).
\end{aligned}$$

Similarly, one can check that $I_A \otimes \rho_B - \rho \geqslant 0$ implies that $\mathrm{Tr}(\rho_B^2) \geqslant \mathrm{Tr}(\rho^2)$. Therefore, the trace inequality criterion is weaker than the reduction criterion.

Remark. Although the criterion of Theorem 2.1 is weaker than the reduction criterion, it is easy to handle for some senses, since its form is simpler than the reduction criterion. The criterion of Theorem 2.1 is a complementarity for detecting entanglement of states.

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